

Boundary of the action of Thompson's group F on dyadic numbers

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Abstract

We prove that the Poisson boundary of a simple random walk on the Schreier graph of action $F \curvearrowright \mathbb{D}$, where \mathbb{D} is the set of dyadic numbers in $[0, 1]$, is non-trivial. This gives a new proof of the result of Kaimanovich: Thompson's group F doesn't have Liouville property. In addition, we compute growth function of the Schreier graph of $F \curvearrowright \mathbb{D}$.

1 Introduction

Let G be a group equipped with a probability measure μ . A right random walk on (G, μ) is defined as a Markov chain Z with the state space G and transitional probabilities $\mathbb{P}(Z_{n+1} = g | Z_n = h) = \mu(h^{-1}g)$. Specifying initial measure θ (distribution of Z_0), we obtain a probability measure \mathbb{P}_θ^μ on the space of trajectories $(Z_i)_{i \geq 0} \in G^{\mathbb{Z}_+}$. Usually one takes $\theta = \delta_e$ - Dirac measure at the group identity. The Poisson boundary of the pair (G, μ) can be defined as the space of ergodic components of the time shift on the $(G^{\mathbb{N}}, \mathbb{P}_{\delta_e}^\mu)$ [7]. For more equivalent definitions of the boundary one can look at [6]. A pair (G, μ) is said to have *Liouville property* if the corresponding Poisson boundary is trivial, or, equivalently, when the space of bounded μ -harmonic functions on G is 1-dimensional, i.e. consists of constant functions. A group G has *Liouville property* iff for every symmetric, finitely supported μ the pair (G, μ)

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does. For a recent survey and results on Liouville property and Poisson boundaries see [2], [3] and [4].

In this note we prove that Richard Thompson's group F doesn't have Liouville property. A survey on Thompson's groups is presented in [1]. Here we only mention that question of amenability of F is one of the major open problems now.

2 Main results

Consider a simple random walk on a locally finite graph $G = (V, E)$. Fix a starting point x_0 . This enables trajectory space $V^{\mathbb{Z}_+}$ with a probability measure P . The notion of the boundary is easily adapted to this case: it is the space of ergodic components of the time shift on the $(V^{\mathbb{Z}_+}, P)$. We'll use electrical networks formalism as it appears in [5]. Throughout the paper, $d(\cdot, \cdot)$ will denote standard graph distance. Let $B(x, n) = \{y \in V : d(x, y) \leq n\}$ - ball centered at x of radius n . Define also $\partial B(x, n) = \{y \in V : d(x, y) = n\}$.

Theorem 2.1. *Suppose a locally-finite graph G is given. Fix any vertex x_0 . Let $\Upsilon(x_0)$ be the set of geodesics starting at x_0 . Define $gd(x, n) = \#\{\gamma = [x_0, \dots, x_m] \in \Upsilon(x_0) : x \in \gamma \text{ and } d(x, x_m) = n\}$. Suppose there exist some real numbers $c, C > 0$ and $q > 1$ such that the following conditions are satisfied:*

$$gd(n, x) \leq Cq^n \text{ for every } x \in X \quad (1a)$$

$$cq^n \leq gd(x_0, n) \quad (1b)$$

Then $cap(x_0) > 0$.

Remark 2.2. *For example, it's easy to see that conditions(1) are obviously satisfied for a regular m -tree, with $q = m$.*

Proof. We have to show that if $f \in D_0(N)$, $f(x_0) = 1$ then it's Dirichlet norm is bounded from below by some positive constant. Let n be such that $\text{supp } f \subseteq B(x_0, n - 1)$. All resistances are equal to 1 in our case, so we may

write

$$\begin{aligned}
D(f) &= \sum_{e \in E} (f(e^+) - f(e^-))^2 \geq \sum_{k=0}^{n-1} \sum_{\substack{x \in \partial B(x_0, k) \\ y \in \partial B(x_0, k+1) \\ (x, y) \in E}} (f(x) - f(y))^2 = \\
&= \sum_{\substack{\gamma \in \Upsilon(x_0) \\ [x_0, \dots, x_n] = \gamma}} \sum_{k=0}^{n-1} \frac{(f(x_k) - f(x_{k+1}))^2}{\text{gd}(x_{k+1}, n - k - 1)} \geq \sum_{\substack{\gamma \in \Upsilon(x_0) \\ [x_0, \dots, x_n] = \gamma}} \frac{(\sum_{k=0}^{n-1} f(x_k) - f(x_{k+1}))^2}{\sum_{k=0}^{n-1} \text{gd}(x_{k+1}, n - k - 1)} \geq \\
&\geq \text{gd}(x_0, n) \frac{1}{\sum_{k=0}^{n-1} C q^{n-k-1}} \geq c q^n \frac{1}{\sum_{k=0}^{n-1} C q^{n-k-1}} \geq \frac{c(q-1)}{C}
\end{aligned}$$

For the first inequality, we cancel edges which connect vertices which connect vertices at the same distance from x_0 . For the second equality, we consider geodesics from x_0 to points at the distance n and sum quantities $\frac{(f(x_k) - f(x_{k+1}))^2}{\text{gd}(x_{k+1}, n - k - 1)}$ over them, getting $(f(x_k) - f(x_{k+1}))^2$ by definition of gd . We use next Cauchy-Schwartz and finiteness of support of f : $f \equiv 0$ outside of $B(x_0, n - 1)$. Thus we have

$$D(f) \geq \frac{c(q-1)}{C}$$

for every finitely-supported f , hence $\text{cap}(x_0) > 0$ \square

This implies, by theorem (2.12) from [5], that simple random walk on G is transient. Now we are going to establish a theorem which connects transience of certain random walks to non-triviality of boundary. Following [6], we call subset $A \subset G$ a trap, if $\lim_n \mathbb{1}(Z_n \in A)$ exists for almost all trajectories $Z \in G^{\mathbb{N}}$. We call a graph transient if the simple random walk on it is transient.

Theorem 2.3. *Let T be a tree with a root vertex v such that for each descendant v_1, \dots, v_n of v ($n \geq 2$) a subtree T_i rooted at v_i is transient. Then the boundary of simple random walk on T is nontrivial.*

Proof. Take any T_i . Almost surely, every trajectory hits v only finitely many times. The only way to move from T_i to T_j is to pass by v . Therefore, for any i , we'll stay inside or outside of T_i from some moment. This means that T_i is a trap. Let's prove that it is nontrivial, i.e. random walk will stay at T_i

with positive probability. If this is true for each i , then every T_i , $1 \leq i \leq n$, is a nontrivial trap, so boundary is indeed nontrivial. Consider the following set of trajectories of the simple random walk on T :

$$A = \{Z : Z_1 = v_i, \forall k \geq 2 \ Z_k \neq v_i\}.$$

In addition, consider the set of trajectories of the simple random walk on T_i :

$$A' = \{Z' : Z'_0 = v_i, \forall i \geq 1 \ Z'_i \neq v_i\}.$$

Collecting the following facts:

- simple random walk on T goes to v_i with probability $1/n$;
 - probability of going from v_i not to v is $\frac{\deg(v_i)-1}{\deg(v_i)}$;
 - (Z_{k+1}) $_{k \geq 0} \in A'$, and transition probabilities are the same for Z_{i+1} and Z'_i for $i \geq 1$
- we obtain

$$\mathbb{P}(A) = \frac{1}{n} \frac{\deg(v_i) - 1}{\deg(v_i)} \mathbb{P}(A').$$

This shows us that indeed $\mathbb{P}(A) > 0$, as $\mathbb{P}(A') > 0$ due to transience. \square

Proposition 2.4. *Let H be a graph which is formed by adding a set of graphs G_v with pairwise disjoint sets of vertices to each vertex v in T . Then the boundary of simple random walk on H is nontrivial.*

Proof. Boundary of T is nontrivial, so we have non-constant bounded harmonic function h on T . We can extend it to the whole H by setting

$$\hat{h}(x) = \begin{cases} h(x), & \text{if } x \in T \\ h(v), & \text{if } x \in G_v \end{cases}$$

This way we get non-constant bounded harmonic function \hat{h} on H , so boundary is non-trivial. \square

Recall that Richard Thompson's group F is defined as the group of all continuous piecewise linear transformations of $[0, 1]$, whose points of non-differentiability belong to the set of dyadic numbers and derivative, where it exists, is an integer power of 2. It is known to be 2-generated. Now we are ready to prove the main theorem.

Theorem 2.5. *Thompson's group F does not have Liouville property.*

works. Hence, we can apply consequently 2.1, 2.3 and 2.4 to see that there are non-constant bounded harmonic functions on \mathcal{H} , so, by the remark in the beginning of the proof, on the Thompson's group F . \square

Remark 2.6. *The fact that simple random walk on the Thompson's group F has nontrivial boudary is first proven by Kaimanovich in [9].*

3 Growth function of \mathcal{H}

In [10] different types of growth functions for groups are defined. We adapt these definitions to Schreier graphs of group actions. We'll compute growth function of \mathcal{H} . Suppose we have a Schreier graph of action of a group G on set X . Fix some starting point $p \in X$. A cone type of a vertex x is defined as follows:

$$C(x) = \{g \in G : \text{if } w \text{ is a geodesic from } p \text{ to } x, \text{ then } wg \text{ is a geodesic from } p \text{ to } g(p)\}.$$

Complete geodesic growth function is defined as

$$L(z) = \sum_{g \in G: g \text{ is a geodesic for } g(p)} gz^{|g|}.$$

Geodesic growth function is defined by sending all group elements to 1, namely

$$l(z) = \sum_{g \in G: g \text{ is a geodesic for } g(p)} z^{|g|}.$$

Orbit growth function is defined as

$$\widehat{l}(z) = \sum_{n=0}^{\infty} \#\{x \in X : \exists g \in G - \text{geodesic} : |g| = n, g(p) = x\} z^n.$$

Now consider \mathcal{H} . We are interested in geodesics starting at point $p = 1/2(100\dots)$. Then we have 5 cone types of vertices:

- Type 0: point 1 (which corresponds to $1/2$);
- Type 1: black vertices;
- Type 2: grey vertices excluding 1;
- Type 3: white vertices on the tree;

·Type 4: white vertices not on the tree.

Let's write $\Lambda_i^n = \sum_{g \in C_i, |g|=n} g$, where C_i is the i -th cone type. Then one gets recurrent relations:

$$\begin{aligned}\Lambda_0^n &= \Lambda_1^{n-1}a + \Lambda_3^{n-1}a^{-1} \\ \Lambda_1^n &= \Lambda_1^{n-1}a \\ \Lambda_2^n &= \Lambda_1^{n-1}a + \Lambda_2^{n-1}b + \Lambda_3^{n-1}a^{-1} \\ \Lambda_3^n &= \Lambda_2^{n-1}b + \Lambda_4^{n-1}(a^{-1} + b^{-1}) \\ \Lambda_4^n &= \Lambda_4^{n-1}(a^{-1} + b^{-1})\end{aligned}\tag{2}$$

Denote L_i^n the number of geodesics of length n starting from a vertex of type i , leading to different points, i.e. $L_i^n = \partial B(x_i, n)$ for x_i being a vertex of type i . Then recurrent relations are:

$$\begin{aligned}L_0^n &= L_1^{n-1} + L_3^{n-1} \\ L_1^n &= L_1^{n-1} \\ L_2^n &= L_1^{n-1} + L_2^{n-1} + L_3^{n-1} \\ L_3^n &= L_2^{n-1} + L_4^{n-1} \\ L_4^n &= L_4^{n-1}\end{aligned}\tag{3}$$

Let $\Lambda^n = (\Lambda_0^n, \Lambda_1^n, \Lambda_2^n, \Lambda_3^n, \Lambda_4^n)^T$ and $L^n = (L_0^n, L_1^n, L_2^n, L_3^n, L_4^n)^T$. We compute $\tilde{L}(z) = \sum_{n=0}^{\infty} \Lambda^n z^n$ - extended complete geodesic growth function and $\hat{L}(z) = \sum_{n=0}^{\infty} L^n z^n$ - geodesic orbit growth function (if two geodesics lead to the same point, they are counted as one). By recurrent formulas, we have

$$\tilde{L}(z) = \sum_{n=0}^{\infty} \mathbf{A}^n \Lambda_0 z^n = (I_5 - \mathbf{A}z)^{-1} \tilde{\Lambda}_0 \text{ and } \hat{L}(z) = (I_5 - Bz)^{-1} L_0,$$

where $\tilde{\Lambda}_0 = (e, e, e, e, e)^T$, $L_0 = (1, 1, 1, 1, 1)^T$ and transition matrices A, B, \mathbf{A} are obtained from recurrent relations 2, 3. In particular, geodesic growth function is given by the first coordinate of vector-function

$$\bar{l}(z) = (I_5 - Az)^{-1} \Lambda_0,$$

where $\Lambda_0 = L_0 = (1, 1, 1, 1, 1)^T$. Performing calculations, one gets

$$\bar{l}(z) = \left(\frac{1}{1-2z}, \frac{1}{1-z}, \frac{1}{1-3z+2z^2}, \frac{1}{1-3z+2z^2}, \frac{1}{1-2z} \right).$$

So, $\frac{1}{1-2z}$ is the geodesic growth function for our Schreier graph. Also we get

$$\widehat{L}(z) = \left(\frac{1+z}{1-z-z^2}, \frac{1}{1-z}, \frac{1+z}{1-2z+z^3}, \frac{1}{1-2z+z^3}, \frac{1}{1-z} \right).$$

Finally, we see that we've obtained $l(z) = \frac{1}{1-2z}$ and $\tilde{l}(z) = \frac{1+z}{1-z-z^2}$.

$$\tilde{l}(z) = \frac{1+z}{1-z-z^2} = \frac{\varphi^2}{\sqrt{5}(1-\varphi z)} - \frac{\hat{\varphi}^2}{\sqrt{5}(1-\hat{\varphi} z)},$$

where $\varphi = \frac{\sqrt{5}+1}{2}$, $\hat{\varphi} = \frac{-\sqrt{5}+1}{2}$. So, $L_n = |\partial B(p, n)| = \frac{\varphi^{n+2} - \hat{\varphi}^{n+2}}{\sqrt{5}}$. $|B(p, n)| = \sum_{k \leq n} L_k = \frac{\varphi^{n+3} - \varphi^2}{\sqrt{5}(\varphi-1)} - \frac{\hat{\varphi}^{n+3} - \hat{\varphi}^2}{\sqrt{5}(\hat{\varphi}-1)}$.

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